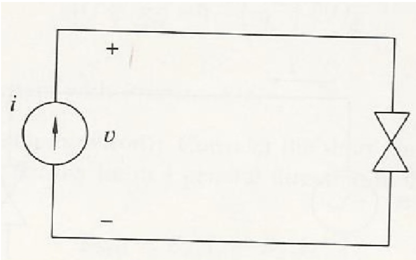
Homework 9

PHYS798C Spring 2024
Due Thursday, 18 April, 2024

1 Current Driven JJ

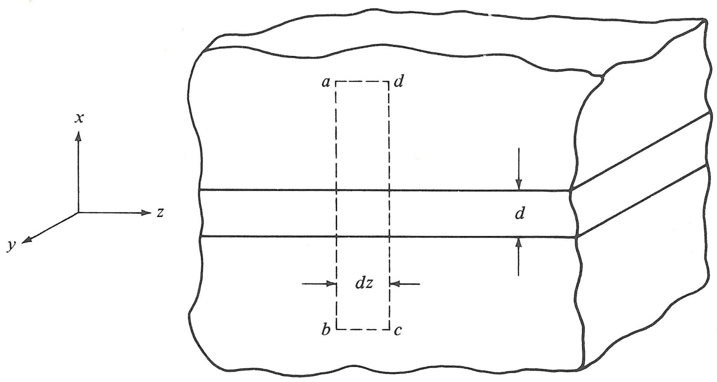
A single basic lumped junction with a critical current I_c is driven by a current source $i(t) = I_{dc} + I_{ac}\cos(\omega_{ac}t)$, as shown in the figure. The dc part of the current is given by I_{dc} and the ac part is at frequency ω_{ac}. The driving current is always less than the critical current, that is, $|I_{dc}| + |I_{ac}| < I_c$.



(a) For this circuit show that, in general, the phase difference across the junction is
$\varphi(t) = \arcsin\left(\frac{I_{dc} + I_{ac}\cos(\omega_{ac}t)}{I_c}\right)$,
and the voltage across the junction is
$$v = \frac{\Phi_0}{2\pi}\frac{1}{\frac{\partial i}{\partial\varphi}}\frac{di}{dt} = -\frac{\Phi_0}{2\pi I_c}\frac{\omega_{ac}I_{ac}\sin(\omega_{ac}t)}{\sqrt{1-\left(\frac{I_{dc}}{I_c}+\frac{I_{ac}}{I_c}\cos(\omega_{ac}t)\right)^2}}.$$
(b) If $I_{dc} >> I_{ac}$, show that the voltage can be approximated by
$$v = -\frac{\Phi_0}{2\pi I_c}\frac{\omega_{ac}I_{ac}\sin(\omega_{ac}t)}{\sqrt{1-\left(\frac{I_{dc}}{I_c}\right)^2}}.$$
(c) If $I_{dc} >> I_{ac}$, show that the voltage can be described in terms of a constant inductance L, which is given by
$$L = \frac{\Phi_0}{2\pi I_c\sqrt{1-\left(\frac{I_{dc}}{I_c}\right)^2}}.$$
(d) In the opposite limit $I_{dc} << I_{ac}$, show that the voltage can be approximated by
$$v = -\frac{\Phi_0}{2\pi I_c}\frac{\omega_{ac}I_{ac}\sin(\omega_{ac}t)}{\sqrt{1-\left(\frac{I_{ac}}{I_c}\cos(\omega_{ac}t)\right)^2}}.$$
(e) Show that the dc component of the voltage is zero as long as $|I_{dc}| + |I_{ac}| < I_c$.

2 Long Josephson Junctions

Consider a pair of identical semi-infinite superconductors sandwiching an insualting material of thickness d. Consider the coordinate system and integration loop shown in the Figure. We will derive two wave equations for electromagnetic waves propagating in this parallel plate waveguide.



(a) Assume that the integration loops go deep enough into the superconductors such that the currents and fields go to zero for the segments $a-d$ and $b-c$ of length dz. Apply Faraday's law $\vec{\nabla}\times\vec{E}=-\frac{\partial\vec{B}}{\partial t}$ on this loop (recall the integral form: $\oint_C \vec{E}\cdot d\vec{l} = -\frac{\partial}{\partial t}\iint_S \vec{B}\cdot d\vec{S}$), and operate with $\partial/\partial z$ to derive the result

$$\frac{\partial^2 E_x^0}{\partial z^2} = -\left(\frac{2\lambda+d}{d}\right)\frac{\partial^2 B_y^0}{\partial t\partial z}$$

where the superscript ^{0} denotes quantities in the insulating layer.

(b) Take a similar integration loop normal to the $z-$axis and derive the result,

$$\frac{\partial^2 E_x^0}{\partial y^2} = \left(\frac{2\lambda+d}{d}\right)\frac{\partial^2 B_z^0}{\partial t\partial y}.$$

(c) Now operate with $\partial/\partial t$ on the $x-$component of Ampere's Law $\vec{\nabla}\times\vec{B}=\mu_0\left(\vec{J}+\varepsilon_0\frac{\partial\vec{E}}{\partial t}\right)$ to obtain the result

$$\frac{1}{\mu}\left(\frac{\partial^2 B_z^0}{\partial y\partial t}-\frac{\partial^2 B_y^0}{\partial z\partial t}\right)=\frac{\partial J_x}{\partial t}+\epsilon\frac{\partial^2 E_x^0}{\partial t^2}$$

(d) Now substitute the derivatives of B^0 from (a) and (b) to derive the result,

$$\left(\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}-\frac{1}{v_{ph}^2}\frac{\partial^2}{\partial t^2}\right)E_x^0=\frac{1}{\varepsilon v_{ph}^2}\frac{\partial J_x}{\partial t}$$

where $v_{ph}=\frac{1}{\sqrt{\varepsilon\mu}}\sqrt{\frac{d}{d+2\lambda}}$. Note that if $\frac{\partial J_x}{\partial t}=0$ then this equation describes TEM waves propagating down a superconducting waveguide, known as Swihart modes. The inductance of the waveguide is proportional to $L\sim d+2\lambda$ and the capacitance $C\sim 1/d$, hence $v_{ph}\sim\frac{1}{\sqrt{LC}}\sim\sqrt{\frac{d}{d+2\lambda}}$. As the penetration depth grows the phase velocity of the wave can be slowed significantly.

(e) Now assume that there is Josephson coupling between the two plates through the insulator. Taking the top plate as the positive potential we have from the second Josephson equation

$$\frac{\partial\gamma}{\partial t}=\frac{2eV}{\hbar}=-\frac{2eE_x^0 d}{\hbar}$$

Solving this for E_x^0 and using the first Josephson equation, derive the result

$$\left(\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}-\frac{1}{v_{ph}^2}\frac{\partial^2}{\partial t^2}\right)\gamma(y,z)=\frac{\sin\gamma(y,z)}{\lambda_J^2},$$

where we have taken the $t = 0$ time reference where $\gamma = 0$ in the time integration and $\lambda_J^2 \equiv \hbar / \left[2eJ_c\mu\left(2\lambda + d\right)\right]$ is the Josephson penetration depth. This is a form of the famous sine-Gordon equation. Estimate the Josephson penetration depth for a long junction with $2\lambda + d = 90\ nm$ and $J_c = 10^2 A/cm^2$.

(f) Consider a linearlized solution to the Josephson wave equation. Assume small phase variation across the junction as a function of time so that $\gamma\left(y, z, t\right) = \gamma_0\left(y, z\right) + \gamma_1\left(y, z, t\right)$, where ϕ_0 is a spatially-dependent time average, and $\gamma_1 \ll \gamma_0$. With this substitution and assuming that $\cos\gamma_1 \simeq 1$, show that

$$\left(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{v_{ph}^2}\frac{\partial^2}{\partial t^2}\right)\gamma_1 = \left(\frac{\cos\gamma_0}{\lambda_J^2}\right)\gamma_1$$

Now assume that γ_0 is a constant. Assume a travelling wave solution of the form $\gamma_1 = e^{-i\left(\omega t - \overrightarrow{\beta}\cdot\overrightarrow{r}\right)}$ and find the dispersion relation $\omega^2 = \beta^2 v_{ph}^2 + \omega_p^2$ where the Josephson plasma frequency is defined as $\omega_p^2 = \left(\frac{v_{ph}}{\lambda_J}\right)^2 \cos\gamma_0$. Plot the dispersion relation for both the Swihart and Josephson modes. How can you tune the Josephson plasma frequency? In the limit $\beta = 0$ there is no rf magnetic field present and there is a periodic exchange of energy between the electric field and the Josephson coupling energy, in close analogy with cold plasma oscillations.